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Role of Dilations in Diffeomorphism-Covariant Algebraic Quantum Field Theory

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A generalization of algebraic quantum field theory on differentiable manifolds is given in terms of nets of ∗-algebras over open sets of the manifold. The present investigations are motivated by diffeomorphism invariance and finite localization as they appear, e.g., in quantum gravity. A possible generalization of Haag–Kastler axioms for differentiable manifolds is discussed and a minimal framework based on isotony, covariance, and a state-dependent GNS construction is presented. Possible adaptions of Haag's commutant duality are discussed in a specific setting of one-parameter families of finite and nondegenerate nested localization domains of the net, with universal minimal and maximal algebras for the small and large limits of the net, respectively. For von Neumann algebras the modular group is discussed. The geometric interpretation of a one-parameter subgroup of outer isomorphisms is related to dilations of the open sets of the net.

1. INTRODUCTION

The following investigations can be seen as an attempt to understand some aspects of quantum field theory (QFT) on differentiable manifolds. This is indeed also a very promising approach to quantum general relativity or (loop) quantum gravity [1]. The quantum analog of general relativistic geometry should be implementable on smooth manifolds without an *a priori* metric structure, the kinematical covariance group being represented by diffeomorphisms. Hence it is useful to generalize the setting of algebraic quantum (field) theory (QFT) such that it becomes a framework for local quantum physics which can be applied also to quantum gravity.

In local quantum physics, observation procedures represent the abstract kinematical framework for possible preparations of measurements, while the

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observables encode the kinds of questions one can ask of the physical system. The covariance group of the observation procedures reflects a general (*a priori*) redundancy of their mathematical implementation. The more sophisticated is the structure of the observation procedures, the smaller the covariance group will be in general. In a concrete observation the kinematical covariance will be broken. So, e.g., in a treelike network of string world tubes a concrete local observation requires the explicit selection of one of many *a priori* equivalent vertices, whence it breaks the covariance which holds for the network of vertices as a whole [2]. However, irrespective of the loss of covariance in a concrete observation (e.g., by the particular frame attached to the observer), the action of the covariance group may still be well defined on the observation procedures. In any case, the loss of covariance in a concrete observation is related to a specific structure of the state of the physical system.

Previous attempts [3, 4] to implement kinematical general covariance and its dynamical breakdown in the spirit of an algebraic, constructive approach [5] to quantum (field) theory have been continued recently [6–8]. The principle of locality is kept by demanding that observation procedures correspond to possible preparations of localized measurements in bounded regions. Note that there is no *a priori* notion of either a metric or a time, nor even a causal structure. Then, on different regions there will be no *a priori* causal relations between observables.

For a net of subalgebras of a Weyl algebra, it is indeed possible [9] to work with a flexible notion of causality rather than with a rigidly given one. In principle it might be possible to construct the net together with its underlying manifold from a partial order via inclusion of the algebras themselves only [10].

Nevertheless, below we start just from a net of $*$-algebras on a differentiable manifold. On this net, a physical state induces dynamical relations, whence the algebra of observables is covariant just under a certain subgroup of the general diffeomorphism group. This subgroup describes a covariance related to the (dynamically relevant) observables. The present examinations emphasize particular one-parameter subgroups of diffeomorphisms which relate the members of a one-parameter family of contractable, simply connected, bounded open sets nested by inclusion. We refer to such particular diffeomorphisms as dilations. As we will see below, under some conditions (satisfied for a typical QFT) on the type of corresponding localized algebras (in particular, for von Neumann factors of type III_1), a one-parameter family of dilations will yield a one-parameter group of outer automorphisms on the algebras. While on a space-time manifolds the appearance of a one-parameter group of outer automorphisms related to time translations is to be expected, in a diffeomorphism-invariant context it is *a priori* not clear what should be the most canonical way to obtain a one-parameter foliation of a manifold such that its parameter is suitable as a dynamical one, i.e., as a time parameter with the leaves as spatial slices.

Here I pursue a pragmatic local point of view motivated by the typical situation in a cosmological observation. There a time parameter only needs to exist locally on bounded regions of the manifold containing the observer at a fixed point *x*. Observations typically take place in a bounded time interval where each time value labels a spatial boundary given by a cosmological horizon limiting the domain of observations. Each one-parameter foliation between the minimal and the maximal bound provide a natural topological cobordism. For the purpose of this paper the latter will always be trivial, since we assume that the topology of all leaves of the foliation are localization domains of the same structure: open, bounded, connected, simply connected, and contractible. One might try to admit topology changes and explore corresponding topological quantum field theory structures. However, we will not follow this approach here, but rather leave it for elsewhere.

The goal here to examine the algebraic structure of a diffeomorphism-covariant local quantum theory in the described setting where local dilations play a distinguished role.

In the particular case where the manifold carries a Riemannian metric, with a given fixed point *x*, the subsets of a Riemannian manifold defined by constant bounds on the geodesic distance from *x* form such a one-parameter family of nested sets which are mapped into each other by a constant rescalings of the metric.

In Section 2 we review the algebraic axioms for (free) QFT on Minkowski space, in particular under the aspect of their possible generalization in a background-independent setting (in particular, without reference to any *a priori* given metric) with covariance under an admissible group (in particular, consistent with boundary conditions) of diffeomorphisms which in general will be much larger than isometries. Looking at the GNS construction and the covariance condition on the Gelfand ideal, one sees that the selection of the (physical) state determines the dynamical subgroup of diffeomorphisms which leave the Gelfand ideal invariant. This is in some analogy to the Borchers algebra approach sketched in ref. 11.

A general framework for diffeomorphism-covariant local quantum physics is implemented in Section 3 in the form of nets of algebras localized on nested open sets around any interior point of the manifold, covariant under certain admissible diffeomorphisms. The generalized axioms of isotony and covariance together with a state-dependent GNS construction form the backbone of this diffeomorphism-covariant setting. The causality condition can be sharpened to Haag duality [11], which plays a key role in the DHR analysis [12]. Haag duality is the first example of an algebraic commutant duality on a net of von Neumann algebras. Recently [6–8] attempts were made to find an appropriate modification of Haag duality that can be applied in consistency with general covariance on an arbitrarily curved background or in a metric-independent setting.

Section 4 introduces minimal and maximal bounds on the net of (finitely) localized algebras and discusses related parametric dilations for sets and algebras between these bounds.

With a minimal (maximal) domain of localization $\mathbb{O}^x_{s_{\min}}$ ($\mathbb{O}^x_{s_{\max}}$) the regularization can be introduced algebraically at the small (large) scale for fields contained in the corresponding algebras of an isotonic net. At the small end this implies that the localization of the algebras and related field will be only finite. In particular, the fundamental constituents of the theory, rather than being pointlike, can be extended objects localized on nondegenerate open sets.

Then, in a further step we examine some specific versions of an algebraic commutant duality in this setting. It turns out that even a relatively mild version restricts the algebraic structure of the net considerably and implies characteristic features, such as a minimal Abelian center, which are absent in the more common quantum field theories on Minkowski space. On the other hand, it reveals a remote analogy to theories with superselection sectors.

In ref. 6 a commutant duality was implemented within the large-scale limit $n \to \infty$ of discretely parametrized domains $\mathbb{O}_n$, $n \in \mathbb{N}$. Unlike there, in refs. 7 and 8 a von Neumann commutant duality $(\mathcal{R}\mathbb{O}^x_{s_{\min}}) = \mathcal{R}'(\mathbb{O}^x_{s_{\max}})$ was introduced between some minimal and some maximal algebra from a net of von Neumann algebras on continuously parametrized bounded domains $\mathbb{O}^x_s$, $s_{\min} \leq s \leq s_{\max}$, around any point x.

We will see below that those earlier versions have quite fatal implications for the related algebraic quantum (field) theory in question. In contrast to ref. 6, it is considered here to be more natural that both $\mathbb{O}^x_{s_{\min}}$ and $\mathbb{O}^x_{s_{\max}}$ are nonempty, bounded, open sets with boundaries given as a horizon for an observer located outside $\mathbb{O}^x_{s_{\max}}$ (at infinity) and inside $\mathbb{O}^x_{s_{\min}}$ (at x) respectively. Clearly, only those diffeomorphisms are admissible here which map $\partial\mathbb{O}^x_{s_{\max}}$ and likewise $\partial\mathbb{O}^x_{s_{\min}}$ to itself. (Note, however, that there is *a priori* not necessary that they vanish on the horizon.) Clearly, this algebraic implementation of small- and large-scale regularization also proves particularly useful in order not to get trapped in the usual conflict between cutoffs and covariance.

The commutant duality can also be viewed as a lift of the scale duality between the small and the large to the algebraic level.

A consequence of the commutant duality is that separability and cyclicity of the GNS vacuum imply each other. In Section 5 both cyclicity and separability are needed in order to extract the modular structure from the net of von Neumann algebras and to obtain the modular group of algebraic isomorphisms of the net.

Section 6 concludes with a brief discussion of some possible implications of the proposed structure for quantum general relativity and *a posteriori* notions of time and causality.

2. ALGEBRAIC AXIOMS OF QFT ON MINKOWSKI SPACE

In this section we review the algebraic axioms of usual QFT on Minkowski space $\mathcal{M}$. These can be formulated in terms of a net of von Neumann algebras $\mathcal{R}(\mathbb{O})$ on open sets $\mathbb{O} \subset \mathcal{M}$ as follows.

Isotony:

$$\mathbb{O}_1 \subset \mathbb{O}_2 \Rightarrow \mathcal{R}(\mathbb{O}_1) \subset \mathcal{R}(\mathbb{O}_2) \tag{2.1}$$

Additivity:

$$\mathbb{O} = \bigcup_j \mathbb{O}_j \Rightarrow \mathcal{R}(\mathbb{O}) = \Big(\bigcup_j \mathcal{R}(\mathbb{O}_j)\Big)'' \tag{2.2}$$

Causality:

$$\mathbb{O}_1 \perp \mathbb{O}_2 \Rightarrow \mathcal{R}(\mathbb{O}_1) \subset \mathcal{R}(\mathbb{O}_2)' \tag{2.3}$$

Covariance:

$$P \ni g \overset{\exists}{\mapsto} U(g) \in U(P): \quad \mathcal{R}(g)(\mathbb{O}) = U(g)\mathcal{R}(\mathbb{O})U(g)^{-1} \tag{2.4}$$

Spectrum Condition:

$$\mathrm{spec}U(\tau) \subset \overline{V^+}, \qquad \tau \subset P \tag{2.5}$$

Vacuum Vector:

$$\begin{aligned} &\exists \Omega \in \mathcal{H}, \quad \|\Omega\| = 1: \\ &(\text{cyclic}) \quad \Big(\bigcup_{\mathbb{O}} \mathcal{R}(\mathbb{O})\Big)\Omega \overset{\text{dense}}{\subset} \mathcal{H} \\ &(\text{invariant}) \quad U(g)\Omega = \Omega, \quad g \in P \end{aligned} \tag{2.6}$$

In the case of a differentiable manifold M (of arbitrary, not necessarily metric geometry and curvature), the purely topological condition (2.1) will be kept unmodified, an analog of condition (2.4) will be maintained when P is no longer the Poincaré group, but the relevant covariance group consistent with the structure of the manifold, and $U(P)$ its strongly continuous representation on the Hilbert space $\mathcal{H}$ on which the elements of the abstract (von Neumann) algebras are represented as bounded operators.

The existence of a cyclic vector Ω as in (2.6) makes sense also in a more general setting; however, then it is in general no longer unique, but state dependent. Its invariance under the strongly continuous action of $U(P)$ will survive with modified covariance group P.

Condition (2.3) requires a causal structure on the manifold and a corresponding notion of a causal complement $\perp$ on its sets. However, here we do not want to restrict M to carry *a priori* such a structure. Therefore, instead of (2.3) we propose below a modified commutant duality introduced on the boundary of the net. Similarly, the positive-energy condition (2.5), although it might be physically very desirable, will cease to be well defined in the general case: In general there will be no obvious analog of the forward light cone V^+. Moreover, already for a general Lorentzian manifold, the covariance group in general need not contain an analog of the translational subgroup $\tau \subset P$.

The additivity condition (2.2) can be written down also in the general case. When we deal with von Neumann algebras, $(\mathcal{R}(\mathcal{O}))''$ is algebraically just the weak $*$-closure of $\mathcal{R}(\mathcal{O})$. However, the sense of this axiom is more, namely to relate this algebraic closure to the causal closure of the sets, which in the standard Minkowski case holds if we choose the index sets $\mathcal{O}$ of the net to be double cones. In this case (2.2) implies the Reeh–Schlieder property, ensuring that the vacuum vector is cyclic and separating [11]. However in the general case without causal structure we have no notion of double cones and it is not at all obvious that (2.2) should be imposed.

In the following section we discuss in more detail those axioms and properties which make perfect sense on a general differentiable manifold without any reference to a background metric or causal structure.

3. COVARIANT NETS OF ALGEBRAS

Given a differentiable manifold M (connected, orientable, Hausdorff, and of finite dimension $\dim M > 2$ in order to avoid pathological cases) a collection $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in M}$ of $*$-algebras $\mathcal{A}(\mathcal{O})$ on bounded open sets $\mathcal{O} \in M$ is called a *net of $*$-algebras* iff

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \tag{3.1}$$

The net is sometimes also denoted by $\mathcal{A} := \cup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$. Self-adjoint elements of $\mathcal{A}(\mathcal{O})$ may be interpreted as possible measurements in $\mathcal{O}$.

A net of algebras on M is Diff(M)-*covariant* if it reflects the covariance of the underlying manifold M under the group of its admissible diffeomorphisms Diff(M). Diff(M) then acts by algebraic isomorphisms on $\mathcal{A} := \cup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$, i.e., each diffeomorphism $\chi \in$ Diff(M) induces an algebraic isomorphism α_χ such that

$$\alpha_\chi\,(\mathcal{A}(\mathbb{O})) = \mathcal{A}(\chi(\mathbb{O})) \tag{3.2}$$

Two sets $\mathbb{O}_1$ and $\mathbb{O}_2$ related by a topological isomorphism (e.g., a diffeomorphism) χ such that $\chi(\mathbb{O}_1) = \mathbb{O}_2$ may be identified straightforwardly only if there are no further obstructing relations between them. A relation like $\mathbb{O}_1 \subset \mathbb{O}_2$, in addition to the previous one, implies that $\mathbb{O}_1$ and $\mathbb{O}_2$ have to be considered as topologically isomorphic, though nonidentical, sets. A similar situation holds on the level of algebras. Isotony (3.1) in connection with covariance (3.2) implies that $\mathcal{A}(\mathbb{O}_1)$ and $\mathcal{A}(\mathbb{O}_2)$ are isomorphic, but nonidentical algebras. Therefore it was a misleading abuse of terminology in previous papers [7, 8] to call α_χ an algebraic automorphism (as, e.g., in ref. 11), although the situation is more complicated in general. In the following, algebras related simultaneously by isotonic inclusion and an algebraic isomorphism are more correctly called just isomorphic rather than automorphic algebras.

The state of a physical system is mathematically described by a positive linear functional ω on $\mathcal{A}$. Given the state ω, one gets via the GNS construction a representation π^ω of $\mathcal{A}$ by a net of operator algebras on a Hilbert space $\mathcal{H}^\omega$ with a cyclic vector $\Omega^\omega \in \mathcal{H}^\omega$. The GNS representation $(\pi^\omega, \mathcal{H}^\omega, \Omega^\omega)$ of any state ω has an associated folium $\mathcal{F}^\omega$, given as the family of those states $\omega_\rho := \text{tr}\ \rho\pi_\omega$ which are defined by positive trace class operators ρ on $\mathcal{H}^\omega$.

A physical state ω implicitly contains all peculiarities of the preparation procedure (e.g., choices of the observer's physical frame, etc.), fixing the ensemble within which the observations of the physical system can be made. Once ω has been specified, one can consider in each algebra $\mathcal{A}(\mathbb{O})$ the equivalence relation

$$A \sim B :\Leftrightarrow \omega'(A - B) = 0, \quad \forall \omega' \in \mathcal{F}^\omega \tag{3.3}$$

These equivalence relations generate the two-sided Gelfand ideal

$$\mathcal{J}^\omega(\mathbb{O}) := \{A \in \mathcal{A}(\mathbb{O}) \,|\, \omega'(A) = 0\} \tag{3.4}$$

in $\mathcal{A}(\mathbb{O})$. The (dynamically relevant) state-dependent algebra of observables $\mathcal{A}^\omega(\mathbb{O}) := \pi^\omega\,(\mathcal{A}(\mathbb{O}))$ may be constructed from the (kinematically relevant) algebra of observation procedures $\mathcal{A}(\mathbb{O})$ by taking the quotient

$$\mathcal{A}^\omega(\mathbb{O}) = \mathcal{A}(\mathbb{O})/\mathcal{J}^\omega(\mathbb{O}) \tag{3.5}$$

The net of state-dependent algebras then is also denoted as $\mathcal{A}^\omega := \cup_{\mathbb{O}}\, \mathcal{A}^\omega\,(\mathbb{O})$. By construction, any diffeomorphism $\chi \in \text{Diff}(M)$ induces an algebraic isomorphism α_χ of the observation procedures. Nevertheless, for a given state ω, the action of α_χ will in general *not* leave $\mathcal{A}^\omega$ invariant. In order to satisfy

$$\alpha_\chi(\mathscr{A}^\omega(\mathbb{O})) = \mathscr{A}^\omega(\chi(\mathbb{O})) \tag{3.6}$$

the Gelfand ideal $\mathscr{I}^\omega(\mathbb{O})$ must transform covariantly, i.e., the diffeomorphism χ must satisfy the condition

$$\alpha_\chi(\mathscr{I}^\omega(\mathbb{O})) = \mathscr{I}^\omega(\chi(\mathbb{O})) \tag{3.7}$$

for some algebraic isomorphism α_χ. Due to nontrivial constraints (3.7), the (dynamical) algebra of observables constructed with respect to the folium $\mathscr{F}^\omega$ in general no longer exhibits the full Diff(M) symmetry of the (kinematical) observation procedures. The symmetry of the observables is dependent on (the folium of) the state ω. Therefore, the selection of a folium of states $\mathscr{F}^\omega$, induced by the actual choice of a state ω, results immediately in a breaking of the Diff(M) symmetry. The diffeomorphisms which satisfy the constraint condition (3.7) form a subgroup. This effective symmetry group is called the *dynamical group* of the state ω. The α_χ is called a *dynamical* isomorphism (w.r.t. the given state ω) w.r.t. χ if (3.7) is satisfied.

The remaining dynamical symmetry group, depending on the folium $\mathscr{F}^\omega$ of states related to ω, has two main aspects which we have to examine in order to specify the physically admissible states: First, it is necessary to specify its state-dependent algebraic action on the net of observables. Second, one has to find a geometric interpretation for the dynamical symmetry group and its action on M.

If we consider the dynamical group as an *inertial*, and therefore global, manifestation of dynamically ascertainable properties of observables, then its (local) action should be correlated with (global) operations on the whole net of observables. This implies that at least some of the dynamical isomorphisms α_χ are not inner. (For the case of causal nets of algebras it was actually already shown that, under some additional assumptions, the isomorphisms of the algebras are in general not inner [13].)

Note that one might consider instead of the net of observables $\mathscr{A}^\chi(\mathbb{O})$ the net of associated von Neumann algebras $\mathscr{R}^\chi(\mathbb{O})$, which can be defined even for unbounded $\mathscr{A}^\chi(\mathbb{O})$, if we take from the modulus of the von Neumann closure $(\mathscr{A}^\chi(\mathbb{O}))''$ all its spectral projections [3]. Then the isotony (3.1) induces a likewise isotony of the net $\mathscr{R}^\chi := \cup_{\mathbb{O}} \mathscr{R}^\chi(\mathbb{O})$ of von Neumann algebras.

4. LOCAL DILATIONS

In the following I want to exhibit the possibility of introducing simultaneously regularizations of the small and of the large on a net of von Neumann algebras supposed (unless stated otherwise) to satisfy all the axioms and properties of the previous section. This essentially exploits a local partial ordering on the net, which is induced by the isotony property of the finitely localized algebras of the net.

Let us now make use of the given (C^∞) topological structure of M and choose at a given point $x \in M$ a topological basis of nonzero open sets $\mathbb{O}_s^x \ni x$ parametrized by a real parameter s with $0 < s < \infty$, such that

$$s_1 < s_2 \Leftrightarrow \mathrm{cl}(\mathbb{O}_{s_1}^x) \subset \mathbb{O}_{s_2}^x \tag{4.1}$$

and

$$s \to 0 \Leftrightarrow \mathrm{cl}(\mathbb{O}_s^x) \to \{x\} \tag{4.2}$$

The standard inclusion $\mathbb{O}_{s_1}^x \subset \mathbb{O}_{s_2}^x$ (used previously [7]) does not exclude the possibility that $\partial\mathbb{O}_{s_1}^x \cap \partial\mathbb{O}_{s_2}^x \neq \emptyset$. Since, for the following, this would be slightly pathological, condition (4.1) uses here $\mathrm{cl}(\mathbb{O}_{s_1}^x) \subset \mathbb{O}_{s_2}^x$ as a slightly stricter inclusion instead.

Let the parameter s be restricted by $0 < s_{\min,x} < s < s_{\max,x} < \infty$. Exploiting local reparametrization invariance, one may assume

$$s_{\min,x} = s_{\min}, \qquad s_{\max,x} = s_{\max} \qquad \forall x \in M \tag{4.3}$$

without loss of generality. Then, for each $x \in M$, open sets $\mathbb{O}_s^x$ with $s \in\ ]s_{\min}, s_{\max}[$ generate local cobordisms between $\partial\mathbb{O}_{s_{\min}}^x$ and $\partial\mathbb{O}_{s_{\max}}^x$, and the isotony property (3.1) implies that

$$s_{\min} < s_1 < s_2 < s_{\max} \Rightarrow$$
$$\mathcal{R}^\omega(\mathbb{O}_{s_{\min}}^x) \subset \mathcal{R}^\omega(\mathbb{O}_{s_1}^x) \subset \mathcal{R}^\omega(\mathbb{O}_{s_2}^x) \subset \mathcal{R}^\omega(\mathbb{O}_{s_{\max}}^x) \tag{4.4}$$

Here, any diffeomorphism $\mathbb{O}_{s_1}^x \mapsto \mathbb{O}_{s_2}^x$ is a *local dilation* at $x \in M$ from $\mathbb{O}_{s_1}^x$ to $\mathbb{O}_{s_2}^x$. Note that all local dilations which preserve covariance of (4.4) must leave invariant $\mathbb{O}_{\min}^x$ and $\mathbb{O}_{\max}^x$.

Now, a commutant duality relation between the inductive limits given by the minimal and maximal algebras is introduced,

$$\mathcal{R}^\omega(\mathbb{O}_{s_{\min}}^x) = (\mathcal{R}^\omega(\mathbb{O}_{s_{\max}}^x))' \tag{4.5}$$

where $\mathcal{R}'$ denotes the commutant of $\mathcal{R}$ within some $\mathcal{R}_{\max} \supset \mathcal{R}$. Then the bicommutant theorem ($\mathcal{R}'' = \mathcal{R}$) implies that likewise also

$$\mathcal{R}^\omega(\mathbb{O}_{s_{\max}}^x) = (\mathcal{R}^\omega(\mathbb{O}_{s_{\min}}^x))' \tag{4.6}$$

If one now demands that all maximal (or all minimal) algebras are isomorphic to each other, independent of the choice of x and the open set $\mathbb{O}_{s_{\max}}^x$ (resp. $\mathbb{O}_{s_{\min}}^x$), then by (4.5) [resp. (4.6)], also all minimal (resp. maximal) algebras are isomorphic to each other. The isomorphism class is then an abstract universal minimal, resp. maximal, algebra, denoted by $\mathcal{R}^{\omega}_{\min}$ and $\mathcal{R}^{\omega}_{\max}$, respectively.

If, as in the following, the commutant is always taken within $\mathcal{R}^{\omega}_{\max}$, the duality (4.5) implies that $\mathcal{R}^{\omega}_{\min}$ is Abelian. (Note, however, that one should also keep in mind the possibility to take the commutant w.r.t. to some larger algebra $\mathcal{R}^{\omega}_{B} \supset \mathcal{R}^{\omega}_{\max}$. Such a choice would possibly include further correlations outside the observable range; it is not considered further here.)

By isotony and (4.1) together with (4.3), the mere existence of $\mathcal{R}^{\omega}_{\min}$, resp. $\mathcal{R}^{\omega}_{\max}$, implies the existence of nontrivial sets $\mathbb{O}^{x}_{s_{\min}}$, resp. $\mathbb{O}^{x}_{s_{\max}}$, at any $x \in M$. By (4.3) we already gauged the size of all these sets to $s_{\min}$, resp. $s_{\max}$, i.e., to a common size (as measured by the parameter s) independent of $x \in M$. So in this case $s_{\min}$ and $s_{\max}$ really denote universal small, resp. large, scale cutoff. Note that, in the context of Section 3, the universality assumption (4.3) is indeed nontrivial because local diffeomorphisms consistent with the structure above must preserve $s_{\min}$, $s_{\max}$, and the monotony of the ordered set $]s_{\min}, s_{\max}[$. The number $s \in\,]s_{\min}, s_{\max}[$ parametrizes the partial order of the net of algebras spanned between the inductive limits $\mathcal{R}^{\omega}_{\min}$ and $\mathcal{R}^{\omega}_{\max}$.

Note that these inductive limits are, strictly speaking, not part of the diffeomorphism-invariant net itself. In particular, the minimal Abelian center $\mathcal{R}^{\omega}_{\min}$ should be exempted because otherwise, by isotony and dilation covariance, all algebras would be isomorphic to the Abelian center, whence we would deal with a classical rather than a quantum theory. By duality, then, $\mathcal{R}^{\omega}_{\max}$ should likewise be exempted from the net itself.

Although in local QFT usually the support of an algebra and that of its commutant are not at all related, it might be nevertheless instructive to consider for the moment a net which is not necessarily covariant under dilations (otherwise this remark would again refer only to the trivial Abelian case), and which has the property that (sufficiently large) algebras of the net satisfy

$$(\mathcal{R}^{\omega}(\mathbb{O}^{x}_{s}))' \subset \mathcal{R}^{\omega}(\mathbb{O}^{x}_{s}) \tag{4.7}$$

Then, with the center of $\mathcal{R}^{\omega}(\mathbb{O}^{x}_{s})$ defined as $\mathfrak{Z}(\mathcal{R}^{\omega}(\mathbb{O}^{x}_{s})) := \mathcal{R}^{\omega}(\mathbb{O}^{x}_{s}) \cap (\mathcal{R}^{\omega}(\mathbb{O}^{x}_{s}))'$, one obtains for the net $\mathfrak{Z}(\mathcal{R}^{\omega}(\mathbb{O}^{x}_{s})) = (\mathcal{R}^{\omega}(\mathbb{O}^{x}_{s}))' = \mathfrak{Z}((\mathcal{R}^{\omega}(\mathbb{O}^{x}_{s}))')$, and correspondingly for the inductive limit $\mathfrak{Z}(\mathcal{R}^{\omega}_{\max}) = \mathcal{R}^{\omega}_{\min} = \mathfrak{Z}(\mathcal{R}^{\omega}_{\min})$. So, for a pair of commutant dual algebras satisfying Eq. (4.7), the smaller one is always Abelian, namely it is the center of the bigger one. With (4.7), the isotony of the net implies the existence of an algebra $\mathfrak{Z}^{\omega}$ which is *maximal Abelian*, in other words, commutant self-dual, satisfying $\mathfrak{Z}^{\omega} = (\mathfrak{Z}^{\omega})' = \mathfrak{Z}(\mathfrak{Z}^{\omega})$. This algebra is given explicitly via the Abelian net of all centers, $\mathfrak{Z}^{\omega} := \cup_{\mathbb{O}}\, \mathfrak{Z}(\mathcal{R}^{\omega}(\mathbb{O}))$. The algebra $\mathfrak{Z}^{\omega}$, located on an underlying set $\mathbb{O}^{x}_{s_z}$ of intermediate size $s_{\min} < s_z < s_{\max}$, separates the small Abelian algebras $\mathcal{R}^\omega(\mathcal{O}_s^x) = \mathcal{Z}(\mathcal{R}^\omega(\mathcal{O}_s^x))$, with $s \leq s_z$, from larger non-Abelian algebras $\mathcal{R}^\omega(\mathcal{O}_s^x) = (\mathcal{Z}(\mathcal{R}^\omega(\mathcal{O}_s^x)))'$, with $s > s_z$.

For a net subject to (4.7), its lower end is Abelian, whence observations on small regions with $s \leq s_z$ are expected to be rather classical. Nevertheless, for increasing size $s > s_z$, there might well exist a nontrivial quantum (field) theory (in fact, it was shown [13] that, for causal nets, the algebras of QFT are not Abelian and not finite-dimensional) if dilation covariance does not hold on the whole net. However, common sense might well justify rejecting this possibility as unphysical.

It was speculated [14] also that there might be kinetic substructure of quantum general relativity. There may be a large, well-defined class of elementary constituents being purely classical, thus yielding finitely localized Abelian algebras. We note here that the Abelian algebra of free loops in quantum general relativity provides indeed such classical constituents [1, 15].

Nevertheless, the following investigations all hold independent of relation (4.7). Indeed, we will see below that (4.7) could only make sense if we take the commutant w.r.t. some algebra essentially larger than $\mathcal{R}^\omega_{\max}$.

5. MODULAR STRUCTURE AND DILATIONS

If we consider the small- and large-scale cutoffs as introduced above, it should be clear that only regions of size $s \in \,]s_{\min}, s_{\max}[$ are admissible for measurement. Again we consider finitely localized von Neumann algebras which (unless stated otherwise) are supposed to satisfy all axioms and properties of Section 3. The commutant duality between $\mathcal{R}^\omega_{\min}$ and $\mathcal{R}^\omega_{\max}$ inevitably yields large-scale correlations in the structure of any physical state ω on any admissible region $\mathcal{O}_s^x$ of measurement at x. Let us assume here that ω is properly correlated, i.e., the GNS vector Ω^ω is already cyclic under $\mathcal{R}^\omega_{\min}$. Then, by duality, it is separating for $\mathcal{R}^\omega_{\max} = \mathcal{R}^{\omega\prime}_{\min}$. Furthermore, Ω^ω is also cyclic under $\mathcal{R}^\omega_{\max}$, and hence separating for $\mathcal{R}^\omega_{\min}$.

So Ω^ω is a cyclic and separating vector for $\mathcal{R}^\omega_{\min}$ and $\mathcal{R}^\omega_{\max}$, and by isotony also for any local von Neumann algebra $\mathcal{R}^\omega(\mathcal{O}_s^x)$.

As a further consequence, on any region $\mathcal{O}_s^x$, the Tomira operator S and its conjugate F can be defined densely by

$$SA\Omega^\omega := A^*\Omega^\omega \qquad \text{for} \quad A \in \mathcal{R}^\omega(\mathcal{O}_s^x) \tag{5.1}$$

$$FB\Omega^\omega := B^*\Omega^\omega \qquad \text{for-}B \in \mathcal{R}^\omega(\mathcal{O}_s^x)' \tag{5.2}$$

The closed Tomita operator S has a polar decomposition

$$S = J\Delta^{1/2} \tag{5.3}$$

where J is antiunitary and $\Delta := FS$ is the self-adjoint, positive modular operator. The Tomita-Takesaki theorem [11] provides us with a one-parameter group of state–dependent isomorphisms α_t^ω on $\mathcal{R}^\omega(\mathbb{O}_s^x)$ defined by

$$\alpha_t^\omega(A) = \Delta^{-it} A \, \Delta^{it} \qquad \text{for} \quad A \in \mathcal{R}_{\max}^\omega \tag{5.4}$$

So, as a consequence of commutant duality and isotony assumed above, we obtain here a strongly continuous unitary implementation of the modular group of ω, which is defined by the one-parameter family of isomorphisms (5.4), given as the conjugate action of operators $e^{-it\ln\Delta}$, $t \in \mathbb{R}$. By (5.4), the modular group, for a state ω on the net of von Neumann algebras defined by $\mathcal{R}_{\max}^\omega$, might be considered as a one-parameter subgroup of the dynamical group. Note that, with Eq. (5.2), in general, the modular operator Δ is not located on $\mathbb{O}_s^x$. Therefore, in general, the modular isomorphisms (5.4) are not inner. The modular isomorphisms are known to act as inner isomorphisms iff the von Neumann algebra $\mathcal{R}^\omega(\mathbb{O}_s^x)$ generated by ω contains only semifinite factors (type I and II), i.e., ω is a semifinite trace.

Above we considered concrete von Neumann algebras $\mathcal{R}^\omega(\mathbb{O}_s^x)$, which are in fact operator representations of an abstract von Neumann algebra $\mathcal{R}$ on a GNS Hilbert space $\mathcal{H}^\omega$ w.r.t. a faithful normal state ω. In general, different faithful normal states generate different concrete von Neumann algebras and different modular isomorphism groups of the same abstract von Neumann algebra.

The outer modular isomorphisms form the cohomology group Out $\mathcal{R}$:= Aut$\mathcal{R}$/Inn$\mathcal{R}$ of modular isomorphisms modulo inner modular isomorphisms. This group is characteristic for the types of factors contained in the von Neumann algebra [16]. Per definition, Out$\mathcal{R}$ is trivial for inner isomorphisms. Factors of type III_1 yield Out$\mathcal{R} = \mathbb{R}$.

In the case of thermal equilibrium states, corresponding to factors of type III_1, there is a distinguished one-parameter group of outer modular isomorphisms which is a subgroup of the dynamical group.

For a QFT on Minkowski space this one-parameter subgroup represented by $\{\Delta^{-it}\}_{t\in\mathbb{R}}$ turns out to correspond geometrically to Lorentz boosts. Similar interpretations hold for more general (globally hyperbolic) space-times. In our general situation possibly without metric or causal background there is no well-defined notion of boosts. Nevertheless there still is a one-parameter subgroup waiting for a some geometric interpretation of its parameter. Let us recall that our partial order defined above is parametrized by open intervals $]s_{\min}, s_{\max}[$ for the full net (in the case of (4.7), $]s_z, s_{\max}[$, dilation covariance makes only sense when the net is restricted to the non-Abelian part). Let us view this interval diffeomorphically as $\mathbb{R}^+$. This way, one may consider dilations of the open sets $\mathbb{O}_s^x$ within the open interval as the geometrical interpretation of the positive semigroup $\{\Delta^{-it}\}_{t\in\mathbb{R}^+}$ contained in the one-parameter group of outer modular isomorphisms of thermal equilibrium states.

Though in the Minkowski case this interpretation seems to disagree with the established geometrical interpretation as boosts rather than dilations, there may nevertheless be a way to reconcile these two interpretations. Note that a boost of any point in a Rindler wedge from parameter $-t$ to t traces out a smooth, timelike world line (of a pointlike observer), and there is a double cone which is the just the causal hull of this world line [see also (6.1) below]. This double cone then contains all events which can be both influenced and registered in a measurement between $-t$ and t. Let now the size of the double cone be defined as $|t|$. Then a boost from t_1 to t_2 (for $0 < |t_1|, |t_2| < \infty$) along any such world line is associated with a rescaling of the double-cone size from $|t_1|$ to $|t_2|$ by a dilation of the double cone. Also, conversely the latter dilation defines an element of the positive semigroup of the outer one-parameter modular group. Note also that, for double cones, the partial order can be related to the split property of the algebras [10].

A different, but related physical interpretation of the modular group has been given by the hypothesis [17] of a thermal time. Indeed, in usual QFT, a local equilibrium state might be characterized as a KMS state [11, 18] over the algebra of observables on a (suitably defined) double cone, whence the one-parameter modular group in the KMS condition might be related to the time evolution.

Now, the details of the isotony condition (3.1), in relation to the modular invariance (5.4), allow us rather immediately to draw some further conclusions which have not yet been spelled out in previous investigations [7, 8]. First assume strict isotony, i.e.,

$$s_1 < s_2 \Rightarrow \mathcal{R}^{\omega}(\mathbb{O}^x_{s_2}) \underset{\neq}{\subset} \mathcal{R}^{\omega}(\mathbb{O}^x_{s_2}) \tag{5.5}$$

Covariance w.r.t. local dilations then implies isomorphic algebras $\mathcal{R}^{\omega}(\mathbb{O}^x_{s_1}) \cong \mathcal{R}^{\omega}(\mathbb{O}^x_{s_2})$ for $s_{\min} < s_1 < s_2 < s_{\max}$, whence, in particular, all algebras have the same von Neumann type. Obviously, here the condition (4.7) would lead to a totally Abelian net if the commutant is not taken in a larger algebra $\mathcal{R}^{\omega}_B \supset \mathcal{R}^{\omega}_{\max}$. Therefore (4.7) is not further considered here. With the commutant duality (4.5) w.r.t. $\mathcal{R}^{\omega}_{\max}$ above, $\mathcal{R}^{\omega}_{\min}$ is Abelian. Hence, the net contains non-Abelian algebras (in particular those with type III_1) only if minimal and maximal sets and algebras are excluded from the net. $\mathcal{R}^{\omega}_{\min}$ and $\mathcal{R}^{\omega}_{\max}$ then only exist as inductive limits of a net of isomorphic algebras, while $\partial\mathbb{O}^x_{\min}$, resp. $\partial\mathbb{O}^x_{\max}$, then are horizon-like boundaries of the open manifold supporting the net. Here, only local regions with $s \in]s_{\min}, s_{\max}[$ are admissible for measurement.

Note that in the case where the manifold carries a Lorentzian metric g, the net must be consistent not only with local dilations of open sets, but also with local dilations of the metric, which take the form of conformal transformations $g(x) \mapsto e^{2\phi(x)}g(x)$, with smooth scale field ϕ on M. If g is consistent with $M =]s_{\min}, s_{\max}[\times \Sigma$ (e.g., by global hyperbolicity), then consistency with covariance demands that ϕ is homogeneous on the boundary of the net, i.e., $\phi(s_{\min}, y) \equiv \phi(s_{\min})$ and $\phi(s_{\max}, y) \equiv \phi(s_{\max})$ for all $y \in \Sigma$.

6. DISCUSSION

We presented a minimal setting of algebraic quantum (field) theory on differentiable manifolds, based on a net of $*$-algebras with the axioms of isotony, diffeomorphism covariance, and a state-dependent GNS representation. The (kinematical) covariance group acts via diffeomorphisms on the open sets of the manifold, and via algebraic isomorphisms on the algebras. In general, for a given state the representation of the algebra of observables on the GNS Hilbert space needs to be covariant only under a (dynamical) subgroup of the general diffeomorphism group.

An algebraic implementation of regularizations at the small and at the large was introduced via universal finitely minimal and maximal algebras, between which the finitely localized algebras of the net are spanned.

We discussed possible adaptions of Haag's commutant duality. The stronger version (4.7) does not make sense if the commutant for all algebras of the net is taken w.r.t. $\mathcal{R}^{\omega}_{\max}$, whence all algebras would be Abelian. Even if we drop (4.7), with this choice of commutant, at least $\mathcal{R}^{\omega}_{\min}$ is Abelian.

An alternative possibility would be to take the commutant w.r.t. a larger algebra $\mathcal{R}^{\omega}_{B} \supset \mathcal{R}^{\omega}_{\max}$. Then we may obtain also non-Abelian $\mathcal{R}^{\omega}_{\min}$. This was not followed here because it would introduce additional problems with algebras outside the net which may give rise then to nontrivial superselection structures. Nevertheless, this issue might deserve investigation elsewhere.

Rather than (4.7), only the much milder commutant duality (4.5) was essential to establish the equivalence of the cyclic and separating properties of the GNS vacuum.

On the basis of a cyclic and separating GNS representation, the modular group of the net could be extracted. Assuming von Neumann algebras (with factors) of type III_1, the modular group acts as outer isomorphisms.

We indicate how dilations may be viewed in a geometric interpretation as a positive semigroup contained in the modular group. In the case of Minkowski-space QFT, this interpretation appears to be consistent with the geometric interpretation of the modular group as Lorentz boosts in the Rindler wedge. Therefore one might hope that conversely in a more general diffeomorphism-invariant setting the action of the modular group will hint toward the natural choices for time and causality. Since the positive semigroup is related to dilations and the partial order of the net, it is plausible that time and dilations are related, too. If the state under consideration is a local equilibrium state, then, just as in usual Minkowski QFT a thermal time may be obtained by the boosts of the modular group, in the more general setting one might obtain such a notion of time from the dilations of the open sets. For any $x \in M$, the parameter s measures the extension of the sets $\mathbb{O}_s^x$. As accessibility regions for a local measurement in M, these sets naturally increase with time. Hence it is natural to suggest that the parameter s might be related to a (thermal) time t such that, for any set $\mathbb{O}_s^x$, $s > s_{\text{min}}$, we have $t < s$ within the set and $t = s$ on the boundary $\partial\mathbb{O}_s^x$.

This thermal time is then related to growing nonidentical algebras of increasing support. If covariance is kept as a condition, these algebras are nevertheless all isomorphic. A nonisomorphic growth of algebras would require releasing the covariance condition, and the growth of abstract (isomorphism classes) of algebras may be used to define an arrow of time.

For the ultralocal limit $s_{\text{min}} \to 0$ (corresponding to usual QFT), it is possible to construct the causal structure for a space-time from the corresponding net of operator algebras [19]. Let us consider here the (*a priori* given) underlying manifold M of the net. Locally around any point $x \in M$ one may induce open double cones as the pullback of the standard double cone which is the conformal model of Minkowski space. These open double cones then carry natural notions of time and causality, which are preserved under dilations. Therefore it seems natural to introduce locally around any $x \in M$ a causal structure and time by specializing the open sets to be open double cones $\mathcal{K}_s^x$ located at x, with timelike extension $2s$ between the ultimate past event p and the ultimate future event q involved in any measurement in $\mathcal{K}_s^x$ at x (time s between p and x, and likewise between x and q). Since the open double cones form a basis for the local topology of M, we might indeed consider equivalently the net of algebras located on open sets

$$\mathbb{O}_s^x := \mathcal{K}_s^x \tag{6.1}$$

Although some (moderate form of) locality might be indeed an indispensable principle within any reasonable theory of observations, it is nevertheless an important, but difficult question whether, and, if at all, under which consistency conditions, a local notion of time and causality might be extended from nonzero local environments of individual points to global regions. This is of course also related to the nontrivial open question of how open neighborhoods of different points $x_1 \neq x_2$ should be related consistently. A final answer to these questions has not been found. At least it seems natural that, on manifolds with no causal relations (like pure space without time), the net should satisfy a disjoint compatibility condition,

$$\mathbb{O}_1 \cap \mathbb{O}_2 = \emptyset \Rightarrow [\mathcal{A}(\mathbb{O}_1), \mathcal{A}(\mathbb{O}_2)] = 0 \tag{6.2}$$

This condition is, e.g., also satisfied for Borchers algebras (see also Ref. 11). Of course the inverse of (6.2) is not true in general.

In a more radical approach some of the above-mentioned difficulties might be avoided by abandoning the notion of points of the manifold and replacing them by more abstract algebraic concepts (which is in fact the spirit of noncommutative geometry). Though we did not go so far here, we nevertheless abandoned points as localization domains. We restricted the net to finitely localized algebras and introduced the algebraic regularization of the small and large end of the net. It is interesting to note here that loop quantum gravity and string theory both come indeed along with basic fields which are only only finitely localized, such as Wilson loops or fields localized on p-branes.

Hence, our generalized framework of algebraic QFT is a useful tool in order to compare quantum gravity with usual QFT. The algebraic approach clarifies the analogies and peculiarities. Some of the features of our proposed framework of diffeomorphism-invariant algebraic QFT which may appear strange from the usual QFT point of view nevertheless appear quite naturally for quantum gravity.

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